

Neutrino oscillations in the formal theory of scattering

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 (Dated: March 3, 2013)

Scattering theory in the Gell-Mann and Goldberger formulation is slightly extended to render a Hamiltonian quantum mechanical description of the neutrino oscillations.

I. INTRODUCTION

The concept of neutrino oscillation has become very familiar to physicists during recent years and the subject is discussed in numerous articles. Nevertheless, there does not appear to be a Hamiltonian treatment which proceeds from fundamental quantum-mechanical principles. The present work represents an attempt to fill this deficit starting from the Gell-Mann and Goldberger formulation of scattering theory [1]. There are very few new results, but the interpretation is somewhat new. The theory is limited to scattering in vacuum and does not include matter effects [2, 3].

The neutrino oscillations have been identified as a potential source of information about fundamental aspects of particle theory by Pontecorvo [4]. Bilenky and Pontecorvo provided an initial theoretical analysis of the neutrino oscillation as a subject of its own merits [5]. Kayser [6] introduced wave packets for neutrinos. Rich [7] offered a space-time approach where neutrino emission, propagation, and absorption are treated as a single process. Giunti, Kim, Lee, and Lee [8] further developed an approach based on Feynman diagrams and wave packets. Recently, Akhmedov and Kopp [9] summarized the current theory status.

Davis, Harmer and Hoffman reported a deficit in the flux of neutrinos from the sun in 1968 [10]. Major collaboration reports on measurements of the neutrino oscillation are available from Super-Kamiokande [11], SNO [12], and KamLAND [13]. Most recent results of T2K and MINOS are described in Refs. [14] and [15], respectively.

Section II introduces the required elements of the formal scattering theory. The neutrino oscillations are discussed in Section III, where it is explained how the standard formula emerges in their theory. Section IV concludes the paper. Appendix A explains relevant details of perturbative energy denominators.

II. FORMAL THEORY

The formal scattering theory of Gell-Mann and Goldberger [1] starts from a full Hamiltonian H which is a sum of a free part H_0 and the interaction part H_I . An incoming state $|\Psi_i(t)\rangle$ of energy E_i is gradually built over

time $\tau = \epsilon^{-1}$ according to the formula

$$|\Psi_i(t)\rangle = \epsilon \int_{-\infty}^0 dT e^{\epsilon T} e^{-iH(t-T)} |\Phi_i(T)\rangle \quad (1)$$

$$= e^{-iHt} \frac{\epsilon}{\epsilon + i(H - E_i)} |\phi_i\rangle. \quad (2)$$

In this formula, the state $|\phi_i\rangle$ is an eigenstate of H_0 ,

$$H_0 |\phi_i\rangle = E_i |\phi_i\rangle, \quad (3)$$

so that

$$|\Phi_i(T)\rangle = e^{-iE_i T} |\phi_i\rangle. \quad (4)$$

The norms of states are not changing with time. The probability that the system is in a state $|\Phi_f\rangle$ at time t is

$$\omega_{fi}(t) = \frac{|A_{fi}(t)|^2}{\|\Phi_f\|^2 \|\Psi_i\|^2}, \quad (5)$$

where the amplitude $A_{fi}(t)$ is

$$A_{fi}(t) = \langle \phi_f | \frac{i\epsilon e^{i(E_f - H)t}}{E_i - H + i\epsilon} | \phi_i \rangle. \quad (6)$$

The differential cross section for the transition $i \rightarrow f$ is equal to the transition rate,

$$P_{fi}(t) = \frac{d}{dt} \omega_{fi}(t), \quad (7)$$

divided by the flux of incoming particles.

Gell-Mann and Goldberger argue that for $t \ll \tau$

$$|A_{fi}(t)|^2 \approx |A_{fi}(0)|^2 \quad (8)$$

and they focus on the case of $t = 0$. However, the time t in the neutrino oscillation experiments is typically greater than τ . For example, in the T2K experiment, the initial pion states are generated from a carbon target of size of about 1 m (using energetic protons) and the pions only move about 100 m before H_I nearly certainly turns them into states of neutrinos and muons. This means that τ is shorter than about $\sim 1 \mu\text{s}$. Since the initial neutron is located about 300 km away, the time t must be greater than $300 \text{ km}/c \sim 1 \text{ ms}$. The time t is thus expected to be more than 10^3 times greater than τ .

In addition, the formal scattering theory requires that the linear dimension of the region in which the states are normalized is much greater than τ times the group velocity of the incident particle wave trains. This condition is satisfied in the T2K experiment when the required region contains both Tokai and Kamioka.

To describe scattering in these circumstances, the formal scattering formula needs to be extended to large times t . Using notation

$$\delta_{fi}^*(t) = \frac{i\epsilon}{E_i - E_f + i\epsilon} \langle \phi_f | e^{i(E_f - H)t} | \phi_i \rangle, \quad (9)$$

$$R_{fi}(t, \epsilon) = \langle \phi_f | H_I e^{i(E_f - H)t} \frac{i\epsilon}{E_i - H + i\epsilon} | \phi_i \rangle \quad (10)$$

$$= \langle \Phi_f(t) | H_I | \Psi_i(t) \rangle, \quad (11)$$

one obtains from Eq. (6) that

$$A_{fi}(t) = \delta_{fi}^*(t) + \frac{1}{E_i - E_f + i\epsilon} R_{fi}(t, \epsilon), \quad (12)$$

$$\dot{A}_{fi}(t) = -i R_{fi}(t, \epsilon). \quad (13)$$

In the spirit of [1], one can suggest that Eq. (12) is useful because it exhibits the energy dependence of $A_{fi}(t)$ in the vicinity of $E_f = E_i$. The resulting transition rate is

$$|A_{fi}(t)|^2 = 2 \text{Im} [\delta_{fi}^*(t) R_{fi}(t, \epsilon)] + \frac{2\epsilon}{(E_i - E_f)^2 + \epsilon^2} |R_{fi}(t, \epsilon)|^2. \quad (14)$$

Since in the neutrino oscillation experiments such as T2K the time τ is shorter than t , one cannot send $\epsilon = \tau^{-1}$ to 0 for finite t . Instead, ϵ approximately accounts for the experimental energy uncertainty and determines the width of density in energy of final states f at energy E_i .

The parameter ϵ also smoothes out all contributing amplitudes as functions of energy. As a result of this smoothing, the interference effects in the complete transition rate are described by the standard oscillation formula, see Eqs. (28) and (29). In the remaining part of the article, it is shown how this happens when one approximates the full Hamiltonian H in the exponential factors in Eqs. (9) and (10) by H_0 .

Further analysis will involve specification of H_I . We choose to continue focusing on the example of T2K.

III. NEUTRINO OSCILLATION

In the T2K experiment, neutrino oscillation occurs in the process

$$\pi^+ n \rightarrow \mu^+ \mu^- p, \quad (15)$$

where the initial state consists of a π^+ -meson prepared in Tokai and a neutron, n , located in Kamioka. The final μ^+ is produced in decay of π^+ in Tokai, and μ^- and proton, p , emerge in Kamioka. The distance covered by the intermediate neutrinos, which are created in the interaction responsible for π^+ -decay and annihilated in the interaction that changes n to p and creates μ^- , is practically equal to the distance from Tokai to Kamioka. We assume that the only particle detected in the final state is μ^- . Detection of other particles does not change our conclusions.

Since the dominant interaction processes are $\pi^+ \rightarrow \mu^+ \nu_\mu$ and $n \nu_\mu \rightarrow p \mu^-$ and the neutrinos carry momenta and energies on the order of 1 GeV, the most appropriate interaction Hamiltonian density for obtaining H_I by integration over space is [16–18]

$$\mathcal{H}_I = \frac{G_F}{\sqrt{2}} \cos \vartheta_C \bar{\mu} \gamma^\alpha (1 - \gamma_5) \nu_\mu \bar{p} \gamma_\alpha (1 - g_A \gamma_5) n - i \frac{F_\pi}{\sqrt{2}} \bar{\nu}_\mu \gamma^\alpha (1 - \gamma_5) \mu \partial_\alpha \pi^\dagger + h.c. \quad (16)$$

The resulting H_I is translation invariant and hence conserves three-momentum. The interaction does not conserve the eigenvalues of H_0 , which are sums of energies of the form $E = \sqrt{m^2 + \vec{p}^2}$ for any particle of mass m and three-momentum \vec{p} .

The central issue is that the neutrino states that are created or annihilated by H_I are not eigenstates of H_0 . One can consider H_I at the hadronic level of Eq. (16), or at the level of electroweak bosons that interact with quarks in π^+ and nucleons. In both cases, the neutrino states that are created or annihilated in the interactions with muons, called μ -neutrinos and denoted by ν_μ , are thought to be properly described by a unique combination of (most likely) three quantum fields $\nu_i(x)$, $i = 1, 2, 3$, each of which corresponds to neutrino states that are eigenstate of H_0 and having different masses. In the case of ν_μ ,

$$\nu_\mu(x) = \sum_{i=1}^3 U_{\mu i} \nu_i(x). \quad (17)$$

Recent data on the mixing coefficients, $U_{\mu i}$, can be found in [19].

The standard formula for neutrino oscillation in the quantum process (15) is obtained from Eq. (14) when the Hamiltonian H in the exponential factors in Eqs. (9) and (10) is approximated by H_0 . In addition, one also expands $(E_i - H + i\epsilon)^{-1}$ in Eq. (10) up to second power of H_I . The result is

$$R_{fi}(t, \epsilon) = \langle \phi_f | H_I \frac{e^{i(E_f - H_0)t}}{E_i - H_0 + i\epsilon} H_I | \phi_i \rangle + \dots, \quad (18)$$

where $|\phi_i\rangle = |\pi^+ n\rangle$ and $|\phi_f\rangle = |\mu^+ \mu^- p\rangle$.

There are only two types of intermediate states that contribute to R_{fi} between the operators H_I in Eq. (18). The first type contains a neutrino, μ^+ , and n . The second type contains an anti-neutrino, π^+ , μ^- , and p . Only the first type of intermediate states can lead to a small energy difference in denominator, due to $E_i - H_0 + i\epsilon$. Thus, the transition rate in Eq. (14) is dominated by the contribution from the first type of intermediate states and the second-type contribution can be neglected.

Namely, the energy-denominator associated with an intermediate state with a neutrino of mass m_i is

$$D_i = E_\nu - E_{\nu_i} + i\epsilon, \quad (19)$$

where E_ν denotes the energy transfer from π^+ to n ,

$$E_\nu = E_{\pi^+} - E_{\mu^+}, \quad (20)$$

and the neutrino energy eigenvalue of H_0 is

$$E_{\nu_i} = \sqrt{m_i^2 + \vec{p}_\nu^2}. \quad (21)$$

All eigenvalues ascribed by H_0 to all intermediate states with neutrinos contain E_{ν_i} with one and the same physical momentum transfer \vec{p}_ν irrespective of the neutrino mass. The value of \vec{p}_ν is implied by the three-momentum conservation in the translation-invariant Hamiltonian interaction terms.

While the real part of D_i in Eq. (19) must be on the order of ϵ or smaller according to the density in energy exhibited in Eq. (14), the energy denominators associated with the intermediate states with anti-neutrinos have real parts on the order of $-2E_\nu$. Therefore, the second-type contributions to the transition rate are about $(\epsilon/E_\nu)^2$ times smaller than the first-type, and are neglected.

Consequently, R_{fi} in Eq. (18) is a sum of three amplitudes each of which corresponds to the intermediate state with a virtual neutrino of a different mass. When one evaluates the time derivative of the modulus squared of the sum at time $t = L/c$, one obtains 9 terms which together combine to

$$\begin{aligned} & \left. \frac{d}{dt} |A_{fi}(t)|^2 \right|_{t=L} \\ &= (2\pi)^3 \delta^{(3)}(\vec{p}_\pi + \vec{p}_n - \vec{p}_{\mu^+} - \vec{p}_p - \vec{p}_{\mu^-}) \\ & \times \frac{2\epsilon}{(E_\pi + E_n - E_{\mu^+} - E_{\mu^-} - E_p)^2 + \epsilon^2} |F_\nu|^2 G_\nu \end{aligned} \quad (22)$$

in a self-explanatory fashion. The factor $|F_\nu|^2$ results from H_I . The factor G_ν is

$$G_\nu = \sum_{i,j} |U_{\mu j} U_{\mu i}|^2 e^{i(E_{\nu j} - E_{\nu i})L} G_{ji}, \quad (23)$$

where

$$G_{ji} = \frac{1}{4E_{\nu j} E_{\nu i}} \frac{1}{D_j^*} \frac{1}{D_i}. \quad (24)$$

The energy denominators D_i are different for different values of i if the masses m_i in the eigenvalues E_{ν_i} are different. The differences between the eigenvalues are very small if the mass differences are very small and if the masses are very small in comparison to the physical momentum transfer $|\vec{p}_\nu|$. Monte-Carlo studies of T2K Collaboration suggest that these conditions are satisfied.

Under the condition that

$$\frac{|m_j^2 \pm m_i^2|}{2|\vec{p}_\nu|} \ll \epsilon, \quad (25)$$

one obtains the approximate G_{ji} that does not depend on the neutrino masses. This is shown in Eqs. (A1) and (A2) in Appendix A. Taking the common approximate factor G_{ji} out of the sum of 9 terms, one arrives at

$$G_\nu = \frac{1}{4\vec{p}_\nu^2} \frac{\sum_{i,j} |U_{\mu j} U_{\mu i}|^2 e^{i(m_j^2 - m_i^2)L/(2|\vec{p}_\nu|)}}{(E_\nu - |\vec{p}_\nu|)^2 + \epsilon^2}. \quad (26)$$

This result implies that the rate of counting μ^- in a distant detector depends only on the ratio $L/|\vec{p}_\nu|$ as in the standard neutrino oscillation formula.

Namely, the numerator in G_ν in Eq. (26) is the standard, distance-dependent μ^- -detection probability,

$$P_{\mu \rightarrow \mu}(L) = \sum_{i,j} |U_{\mu j} U_{\mu i}|^2 e^{i(m_j^2 - m_i^2)L/(2|\vec{p}_\nu|)} \quad (27)$$

$$\approx 1 - \sin^2(2\theta_{23}) \sin^2 \frac{\Delta m_{23}^2 L}{4|\vec{p}_\nu|}. \quad (28)$$

More precisely, the ratio of μ^- counting-rates at two different distances between the neutrino detector and a π^+ -source, such as $L_{far} \approx 300$ km and $L_{near} \approx 280$ m in the T2K experiment, is

$$\frac{\left. \frac{d}{dt} |A_{fi}(t)|^2 \right|_{t=L_{far}}}{\left. \frac{d}{dt} |A_{fi}(t)|^2 \right|_{t=L_{near}}} = \frac{P_{\mu \rightarrow \mu}(L_{far})}{P_{\mu \rightarrow \mu}(L_{near})}. \quad (29)$$

This is how the perturbative scattering theory explains the standard neutrino oscillation formula.

IV. CONCLUSION

A simple extension of the Gell-Mann–Goldberger formulation of scattering theory to the case of a long baseline experiment, leads to the standard neutrino oscillation formula that describes the distance-dependent ratio of muon-counting rates via Eq. (29). This conclusion holds provided the condition (25) for a pion-beam preparation time $\tau = \epsilon^{-1}$ is satisfied and one neglects all interaction effects between the initial pion decay and neutrino absorption in a detector. This is formally facilitated by replacing H with H_0 in the exponents in Eqs. (9) and (10).

The oscillation formula is a result of the interference between scattering amplitudes mediated by virtual states with neutrinos of different masses. All these states mediate a transfer of the same physical three-momentum \vec{p}_ν and energy E_ν . However, their contributions to the total amplitude differ as functions of \vec{p}_ν and E_ν . According to Eq. (23), only a sufficiently large ϵ can smooth out the differences between factors G_{ji} and yield Eq. (26), so that the net effect of the interference can be described using the standard oscillation formula.

The condition (25) indicates that the time of preparing the beam of π^+ mesons must be sufficiently short, the neutrino momentum must be sufficiently large, and the neutrino masses sufficiently small for the oscillation to occur in agreement with the standard formula. Otherwise, one might expect deviations, whose details can be deduced from Eq. (23).

Regarding the replacement of H by H_0 in the exponents in Eqs. (9) and (10), one may observe that precise calculations of weak interaction effects following from \mathcal{H}_I of Eq. (16), require understanding of the cutoff dependence that appeared already in Pontecorvo's work [4]. One also ought to consider interactions with matter [2, 3].

It should be stressed that the conclusion concerning the ratio of transition rates does not automatically translate to the ratios of entire cross sections. Evaluation of cross sections includes averaging over incoming and integration over outgoing states and may involve wave packets, density matrices, entanglement, and various experimental cuts. Such evaluation is a formidable task in a fundamental theory. For example, if neutrinos are coupled through intermediate bosons to quarks, and one attempts to evaluate the complete scattering process including the quark structure of hadrons, in addition to the nuclear binding effects, an entire host of additional theoretical issues arise. To be more specific, the Fermi motion of quarks in nucleons, or of nucleons in nuclei, is a subject of study in its own right [20] and currently available level of theoretical analysis could certainly be improved, e.g., see Eq. (2) in [14].

Finally, we note that the Gell-Mann and Goldberger discussion of a connection between the formal scattering theory and S -matrix formalism in the interaction picture involves the limit of $\epsilon \rightarrow 0$. This limit requires that the beam preparation time τ is much longer than the period in which interactions causing scattering may happen. If this condition is not satisfied and one cannot take the limit of $\epsilon \rightarrow 0$, one has to proceed from fundamental quantum-mechanical principles rather than taking advantage of simplified formulae in which $\epsilon \rightarrow 0$.

Appendix A: Energy denominators

Factor G_{ji} in Eq. (24) can be rewritten as

$$G_{ji} = \frac{1}{4E_{\nu_j}E_{\nu_i}} \frac{1}{\left(E_{\nu} - \frac{E_{\nu_j}+E_{\nu_i}}{2}\right)^2 + \left(i\frac{E_{\nu_j}-E_{\nu_i}}{2} + \epsilon\right)^2} \approx \frac{1}{4E_{\nu_j}E_{\nu_i}} \frac{1}{\left(E_{\nu} - \frac{E_{\nu_j}+E_{\nu_i}}{2}\right)^2 + \epsilon^2}, \quad (\text{A1})$$

which is a valid approximation when $|E_{\nu_j} - E_{\nu_i}|/2 \ll \epsilon$. In addition, using condition (25), one obtains

$$G_{ji} \approx \frac{1}{4\vec{p}_{\nu}^2} \frac{1}{(E_{\nu} - |\vec{p}_{\nu}|)^2 + \epsilon^2}, \quad (\text{A2})$$

which does not depend on the neutrino masses and leads to Eq. (26).

Eq. (A1) differs from Eq. (A2) by the terms that in a leading approximation are inversely proportional to the neutrino momentum and directly proportional to the heaviest neutrino mass squared times $2\Delta/(\Delta^2 + \epsilon^2)$, where $\Delta = E_{\nu} - |\vec{p}_{\nu}|$. Therefore, the leading correction to the standard oscillation formula due to using Eq. (26) instead of Eq. (23), may be expected to be smaller than $c\tau/L$, which currently means smaller than 10^{-3} in T2K. In any case, its actual size depends on the pion beam preparation and the Monte Carlo simulations may even average it to nearly 0 due to the varying sign of Δ in data sampling. A special sampling that secures only inclusion of cases with one sign of Δ would be required to facilitate studies of this correction.

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